Math 370 Homework 3 Due Friday 9/21

Notes

If $\mathbf{A}$ is an $n \times m$ matrix, then the *kernel* of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ given by

$$\ker \mathbf{A} = \{\mathbf{v} \in \mathbb{R}^m | \mathbf{A}\mathbf{v} = \mathbf{0}\}.$$

The kernel of $\mathbf{A}$ is also called the *solution space* of the system $\mathbf{A}\mathbf{v} = \mathbf{0}$, for example see O'Neil Theorem 7.12. The dimension of the kernel of $\mathbf{A}$ is called the *nullity* of $\mathbf{A}$.

The *image* of $\mathbf{A}$ is another term for the column space of $\mathbf{A}$. $\text{image}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$. The dimension of the image of $\mathbf{A}$ is called the *rank* of $\mathbf{A}$ (as it is in the textbook).

O'Neil's Theorem 7.12(2) is usually called the *Rank-Nullity Theorem* because with this language, it says for an $n \times m$ matrix $\mathbf{A}$,

$$\text{rank}\,\mathbf{A} + \text{nullity}\,\mathbf{A} = m$$

Geometrically, the Rank-Nullity Theorem is a statement of conservation of dimension. Thinking of $\mathbf{A}$ as taking a vector $\mathbf{v}$ in $\mathbb{R}^m$ and producing the vector $\mathbf{A}\mathbf{v}$ in $\mathbb{R}^n$, the m dimensions of choice for $\mathbf{v}$ must all be accounted for: Some are annihilated by $\mathbf{A}$ (sent to zero) and the rest are still present in the image of $\mathbf{A}$.

Exercises

Chapter 7.2 # 1, 3

Chapter 7.3 # 1-7 odd, find the reduced row echelon form only.

Chapter 7.5 # 1, 3, 5

Chapter 7.4 # 1, 3, 5, 9, 11, 13: For each of these, also find a basis for the kernel of each matrix, find the nullity, and check that the Rank-Nullity theorem is true for that matrix.

Chapter 7.4 # 15

Problem A: Let $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be a rotation matrix and $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the projection matrix from Homework 2. Let $\mathbf{A} = \mathbf{R}_\theta \mathbf{P}$. Find $\ker \mathbf{A}$ and $\text{image}\,\mathbf{A}$ and draw pictures of both $\ker \mathbf{A}$ and $\text{image}\,\mathbf{A}$ for $\theta = 30°$, $45°$, and $60°$.

Problem B: Let $\mathbf{R}_\theta$ and $\mathbf{P}$ be as in Problem A. Let $\mathbf{B} = \mathbf{P}\mathbf{R}_\theta$. Find $\ker \mathbf{B}$ and $\text{image}\,\mathbf{B}$ and draw pictures of both $\ker \mathbf{B}$ and $\text{image}\,\mathbf{B}$ for $\theta = 30°$, $45°$, and $60°$.